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Superstability Estimates for Anharmonic Systems

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We consider an anharmonic crystal described by variables S_x, $x \in \mathbb{Z}^d$, $S_x \in \mathbb{R}$, with one-body interaction $\sim |S_x|^\alpha$ and nearest neighbor (n.n.) two-body interaction $\sim |S_x - S_y|^\beta$. We prove that, for $\Lambda \subset \mathbb{Z}^d$ bounded, $\Delta \subset \Lambda$,

$$\rho_\Lambda(S_\Delta) \leqslant \exp[\delta|\Delta| - \gamma \sum_{x\in\Delta} |S_x|^\alpha - \gamma \sum_{\substack{x,y\in\Delta \\ x,y\,\mathrm{n.n}}} |S_x - S_y|^\beta]$$

where ρ_Λ is the correlation function for the free boundary condition Gibbs state in Λ, $\gamma > 0$ and δ are suitable constants independent of Λ and Δ. This generalizes previous results obtained in the case $\alpha \geqslant \beta$.

KEY WORDS: Statistical mechanics; anharmonic crystal; probability estimates for correlation functions; DLR processes.

1. INTRODUCTION

In this paper we examine a system of unbounded spins S_x, $x \in \mathbb{Z}^d$, $S_x \in \mathbb{R}$. The interaction we consider is the following: there is a self-energy which behaves asymptotically as $|S_x|^\alpha$ and there are nearest neighbor (n.n.) interactions which behave asymptotically as $|S_x - S_y|^\beta$. The free measure is the Lebesgue measure dS_x.

The statistical mechanics of these systems has been studied in Refs. 2 and 3 when $\alpha \geqslant \beta > 0$. Here we will deal with the case $\beta > \alpha > 0$, namely when the interaction energy dominates over the self-energy, for large values of the spins.

If we think of the above as a schematization of an anharmonic crystal ($\alpha, \beta \neq 2$), then some natural questions arise. Does a limiting equilibrium

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state exist? Namely, are the surface effects negligible even for large values of β? What does the limiting measure look like? For instance, can one prove that the larger β is, the smaller is the probability of large differences between nearest spins? More generally, what are the estimates for the correlation functions?

In the next section we will prove that

$$\rho_\Lambda(S_\Delta) \leqslant \exp\left[\delta|\Delta| - \gamma \sum_{x\in\Delta} |S_x|^\alpha - \gamma \sum_{x,y\in\Delta}' |S_x - S_y|^\beta\right]$$

where ρ_Λ is the correlation function for the free boundary condition Gibbs state in Λ, $\gamma > 0$ and δ are suitable constants independent of Λ and Δ, and $\sum'$ denotes the sum over the n.n. spins. From this it follows by standard arguments that the infinite-volume correlations functions exist[1,3] for regular enough boundary conditions, as is briefly recalled in Section 2.

In Section 3 we prove some estimates left over from Section 2. A concluding section then follows.

2. THE SUPERSTABILITY ESTIMATES

We consider the anharmonic crystal given by a set of variables S_x, $x \in \mathbb{Z}^d$, $S_x \in \mathbb{R}$ (for the sake of simplicity we require $S_x \in \mathbb{R}$; generalization to higher dimension spin spaces is straightforward). The one-body interaction is $\phi(S_x)$ and the two-body interaction $\psi(|S_x - S_y|)$ is present only if x and y are n.n. in $\mathbb{Z}^d$.

We assume there are constants $\bar{A}$, $\bar{A}'$, $A > 0$, $\alpha > 0$ such that

$$\bar{A}|S_x|^\alpha - \bar{A}' \leqslant \phi(S_x) \leqslant A(|S_x|^\alpha + 1) \tag{2.1}$$

while the two-body potential is characterized by $\bar{B}$, $\bar{B}'$, $B > 0$ in such a way that

$$\bar{B}|S_x - S_y|^\beta - \bar{B}' \leqslant \psi(|S_x - S_y|) \leqslant B(|S_x - S_y|^\beta + 1) \tag{2.2}$$

We consider a bounded region $\Lambda \subset \mathbb{Z}^d$, and the Gibbs state at volume Λ with free boundary conditions is defined as usual by

$$\mu(dS_\Lambda) = (1/Z_\Lambda)\exp[-V(S_\Lambda)]\, dS_\Lambda \tag{2.3}$$

where

$$S_\Lambda = \{S_x, x \in \Lambda\}, \qquad dS_\Lambda = \prod_{x\in\Lambda} dS_x$$

dS_x is the Lebesgue measure on $\mathbb{R}$; and

$$V(S_\Lambda) = \sum_{x\in\Lambda} \phi(S_x) + \sum_{x,y\in\Lambda}' \psi(|S_x - S_y|)$$

$$Z_\Lambda = \int dS_\Lambda \exp[-V(S_\Lambda)]$$

Z_Λ is well defined because of Eqs. (2.1) and (2.2). Of course ϕ and ψ should be assumed to be Borel-measurable.

Remark. Both the assumptions that the interaction is n.n. and that the free measure is dS_x are too restrictive. We assumed them for the sake of notational simplicity. It would be quite easy to get Theorem 2.1 below also for finite range, with some other conditions on ψ, and to consider free measures that satisfy the following regularity condition. We define a partition of $\mathbb{R}$ into intervals $I_{\lambda,l}$, where λ is the length of the intervals, $l \in \mathbb{Z}$, and λl is the center of the interval $I_{\lambda,l}$. Then an acceptable free measure $\mu(dx)$ is one such that there exist λ, l_0, $\xi > 0$: $\mu(I_{\lambda,l}) \geqslant \xi$, $\forall |l| \geqslant l_0$.

The correlation functions for the model are defined as

$$\rho_\Lambda(S_\Delta) = \frac{1}{Z_\Lambda}\int_{\mathbb{R}^{\Lambda\setminus\Delta}} dS_{\Lambda\setminus\Delta} \exp[-V(S_\Lambda)], \Delta \subset \Lambda \tag{2.4}$$

In the following theorem we give estimates for $\rho_\Lambda(S_\Delta)$ which are uniform in Λ while Δ is kept fixed. These are analogous to Ruelle's superstability estimates,[1,2] which hold in the above conditions whenever $\alpha \geqslant \beta$. Given these estimates, it will be standard[1,3] to prove the existence of the thermodynamic limit both for the pressure and the correlation functions.

We introduce the function ($\Gamma \subset \mathbb{Z}^d$ is bounded)

$$E(S_\Gamma) = \sum_{x\in\Gamma} |S_x|^\alpha + \sum_{x,y\in\Gamma}{}' |S_x - S_y|^\beta \tag{2.4a}$$

and we state the following result:

Theorem 2.1. There exist $\gamma > 0$ and δ such that, if $\Lambda \supset \Delta$ and $\Lambda \subset \mathbb{Z}^d$ is bounded, then

$$\rho_\Lambda(S_\Delta) \leqslant \exp[-\gamma E(S_\Delta) + \delta|\Delta|] \tag{2.4b}$$

Notice that δ and γ depend on ϕ and ψ but do not depend on Δ and Λ.

Proof. The proof follows the lines of Ruelle's,[2] except for the lower bound of Z_Λ, which in this case requires particular care when $\beta > \alpha$. Therefore we will briefly sketch the proof, isolating the technical points in Proposition 3.1, which will be proven in the next section.

The proof proceeds by induction on $|\Delta|$. Therefore we consider the theorem proven for Δ' and we want to prove it for $\Delta = \Delta' \cup \{0\}$, $0 \notin \Delta'$. Of course there is no loss of generality if we identify the new point in Δ with the origin 0 of $\mathbb{Z}^d$. We introduce a sequence of cubes Λ_q,

$$\Lambda_q = \{x \in \mathbb{Z}^d |\ |x| = \max_{1\leqslant i\leqslant d} |x^i| \leqslant q, \quad x = (x^1, \ldots, x^d)\} \tag{2.5}$$

and we only consider $q \geqslant p_0$, where p_0 is a fixed integer satisfying requirements to be given below. We then introduce a partition of $\mathbb{R}^\Lambda$. To simplify the notation, we sometimes consider $\mathbb{R}^\Lambda$ as the set of $S \in \mathbb{R}^{\mathbb{Z}^d}$, $S = (S_x,$

$x \in \mathbb{Z}^d$), such that $S_y = 0$ if $y \notin \Lambda$. The partition we consider is made up by the following disjoint atoms:

$$x_0 = \{S_\Lambda | E(S_{\Lambda_q}) < q^d \log q, \quad \forall q \geqslant p_0\} \tag{2.6}$$

$$\begin{aligned} x_q = \{S_\Lambda | E(S_{\Lambda_{q-1}}) &\geqslant (q-1)^d \log(q-1), \\ E(S_{\Lambda_{q'}}) &< q'^d \log q', \quad \forall q' \geqslant q\}, \qquad q \geqslant p_0 + 1 \end{aligned} \tag{2.7}$$

In evaluating $\rho_\Lambda(S_\Delta)$, we will decompose the integral into a sum of integrals extended to the atoms of the above partition. We first consider

$$\rho_\Lambda{}^0(S_\Delta) \equiv (1/Z_\Lambda) \int_{x_0} dS_{\Lambda\setminus\Delta} \exp[-V(S_\Lambda)] \tag{2.8}$$

We then write

$$V(S_\Lambda) = V(S_{\Lambda\setminus\{0\}}) + V(S_{\{0\}}) + W(S_{\{0\}}|S_{\Delta'}) + W(S_{\{0\}}|S_{\Lambda\setminus\Delta}) \tag{2.9}$$

where, if $\Gamma \cap \Gamma' = \varnothing$,

$$W(S_\Gamma | S_{\Gamma'}) = \sum_{\substack{x \in \Gamma \\ y \in \Gamma'}}{}' \psi(|S_x - S_y|) \tag{2.10}$$

We note that, since S_Λ is in X_0, there exist two positive constants D_1 and D_2, depending only on p_0, such that

$$|W(S_{\{0\}}|S_{\Lambda\setminus\Delta})| \leqslant D_1$$
$$|S_x| \leqslant D_2, \qquad |x| = 1$$

We can now reconstruct the missing integration over dS_0 in Eq. (2.8) in order to obtain $\rho_\Lambda(S_{\Delta'})$. To do this we use the above estimate and Eqs. (2.1) and (2.2):

$$\begin{aligned} \rho_\Lambda{}^0(S_\Delta) &\leqslant \exp[-V(S_{\{0\}}) - W(S_{\{0\}}|S_{\Delta'}) + D_1] \times \frac{1}{Z_\Lambda} \int_{\substack{|S_x| \leqslant D_2 \\ |x| = 1}} dS_{\Lambda\setminus\Delta} \int_{|S_0| \leqslant 1} dS_0 \\ &\qquad \exp\{-V(S_\Lambda) + 2A + 2d[(D_2 + 1)^\beta + 1]B\} \\ &\leqslant \exp[-W(S_{\{0\}}|S_{\Delta'}) - V(S_{\{0\}}) + D_3]\rho_\Lambda(S_{\Delta'}) \end{aligned} \tag{2.11}$$

where D_3 is a suitable large constant.

We now introduce the convention that $\phi(S_x) = 0$ if $x \notin \Lambda$, $\psi(|S_x - S_y|) = 0$ if either x or $y \notin \Lambda$ (remember also that $S_x = 0$ if $x \notin \Lambda$), and $dS_\Gamma = \prod_{x \in \Gamma \cap \Lambda} dS_x$. Then we consider ($\Gamma^c$ is the complement of Γ)

$$\begin{aligned} \rho_\Lambda{}^q(S_\Delta) &= \frac{1}{Z_\Lambda} \int_{x_q} dS_{\Lambda\setminus\Delta} \exp[-V(S_\Lambda)] \\ &= \frac{1}{Z_\Lambda} \int_{x_q} dS_{(\mathring{\Lambda}_q \cup \Delta)^c}\, dS_{\mathring{\Lambda}_q\setminus\Delta} \\ &\qquad \times \exp[-V(S_{\mathring{\Lambda}_q^c}) - V(S_{\mathring{\Lambda}_q}) - W(S_{\mathring{\Lambda}_q}|S_{\mathring{\Lambda}_q^c})] \end{aligned} \tag{2.12}$$

where $\Lambda_q = \mathring{\Lambda}_q \cup \partial\Lambda_q$, $\mathring{\Lambda}_q = \Lambda_{q-1}$.

Now we split the interaction energy as before:

$$W(S_{\mathring{\Lambda}_q}|S_{\mathring{\Lambda}_q^c}) = W(S_{\Delta\cap\mathring{\Lambda}_q}|S_{\Delta\cap\mathring{\Lambda}_q^c}) + W(S_{\Delta\cap\mathring{\Lambda}_q}|S_{\mathring{\Lambda}_q^c\setminus\Delta}) + W(S_{\mathring{\Lambda}_q\setminus\Delta}|S_{\mathring{\Lambda}_q^c}) \tag{2.13}$$

Then, if p_0 is large enough ($q > p_0$),

$$\begin{aligned}\rho_\Lambda{}^q(S_\Delta) \leqslant{} & \exp[-\tfrac{1}{3}V(S_{\Delta\cap\mathring{\Lambda}_q}) - W(S_{\Delta\cap\mathring{\Lambda}_q}|S_{\Delta\cap\mathring{\Lambda}_q^c})] \\ & \times \frac{1}{Z_\Lambda}\int dS_{(\mathring{\Lambda}_q\cup\Delta)^c}\exp[-V(S_{\mathring{\Lambda}_q^c})]\,\chi(\{E(S_{\partial\Lambda_q}) \leqslant 4dq^{d-1}\log q\}) \\ & \times \int dS_{\mathring{\Lambda}_q\setminus\Delta}\exp\left[-\frac{1}{3}\sum_{x\in\mathring{\Lambda}_q\setminus\Delta}(\bar{A}S_x{}^\alpha - \bar{A}') + \frac{1}{3}\sum_{x,y\in\mathring{\Lambda}_q\setminus\Delta}{}' \bar{B}'\right] \\ & \times \exp[-a(q-1)^d\log(q-1) + \bar{A}'|\Lambda_q| + 2d\bar{B}'|\Lambda_q|]\end{aligned} \tag{2.14}$$

$$a = \min(\bar{A}, \bar{B})\tfrac{1}{3}$$

where we have used repeatedly Eqs. (2.1) and (2.2); Eq. (2.7) has also been used and the memory of the region of integration is left in Eq. (2.14) only in the characteristic function χ appearing in the integral on the variables $S_{\Lambda\setminus(\mathring{\Lambda}_q\cap\Delta)}$. Of course the choice of splitting the energy into thirds in Eq. (2.14) is arbitrary.

We now perform the integral over the variables $S_{\mathring{\Lambda}_q\setminus\Delta}$ and we obtain, for a suitable constant D_4,

$$\begin{aligned}\rho_\Lambda{}^q(S_\Delta) \leqslant{} & \exp[-\tfrac{1}{3}V(S_{\Delta\cap\mathring{\Lambda}_q}) - W(S_{\Delta\cap\mathring{\Lambda}_q}|S_{\Delta\cap\mathring{\Lambda}_q^c})] \\ & \times \exp[-\tfrac{1}{2}aq^d\log q + D_4|\Lambda_q|](1/Z_\Lambda)\int dS_{(\mathring{\Lambda}_q\cup\Delta)^c} \\ & \times \exp[-V(S_{(\mathring{\Lambda}_q)^c})]\,\chi(\{E(S_{\partial\Lambda_q} \leqslant 4dq^{d-1}\log q\})\end{aligned} \tag{2.15}$$

To get the analog of Eq. (2.11) we have to insert new integration variables $S_{\mathring{\Lambda}_q\cap\Lambda}$ in Eq. (2.15) in order to reconstruct $\rho_\Lambda(S_{\Delta\cap\mathring{\Lambda}_q^c})$. If $\alpha \geqslant \beta$ this is quite easy and one can proceed as in Ruelle's papers by considering each S_x, $x \in \mathring{\Lambda}_q \cap \Lambda$, in the interval $(-1, +1)$, for example. The interaction energy with the outside spins $S_{\partial\Lambda_q}$ is controlled by the energy $E(S_{\partial\Lambda_q})$ in Eq. (2.15) and so we only lose a term of the form $\exp[c|\Lambda_q|]$, which will be controlled by the convergence factor $\exp[-\frac{1}{2}aq^d\log q]$. When $\alpha < \beta$ the situation is more complicated (for certain values of d, β, α) because one should let the inside spins vary slowly to control the interaction energy ($|S_x - S_y|^\beta$). On the other hand, the energy factor $|S_x|^\alpha$ is building up a large energy if the spins are not made small. In the next section we prove that there is a configuration Z_{Λ_q} such that:

(i) $Z_x = S_x$, $x \in \partial\Lambda_q$.

(ii) There is a constant b such that, if S'_{Λ_q} satisfies

$$|S_x' - Z_x| \leqslant 1, \qquad x \in \mathring{\Lambda}_q$$

then

$$E(S'_{\Lambda_q}) \leqslant bq^d$$

We then insert the new integral

$$\int_{|S_x - Z_x| \leqslant 1} dS_{\mathring{\Lambda}_q \cap \Lambda}$$

By (ii) we can also insert the energy factor $\exp[-V(S_{\mathring{\Lambda}_q})] \exp[-W(S_{\mathring{\Lambda}_q}|S_{\mathring{\Lambda}_q^c})]$ with an $\exp[+c|\Lambda_q|]$ error, so that finally we get from Eq. (2.15)

$$\begin{aligned} \rho_\Lambda{}^q \leqslant \exp[-\tfrac{1}{3}V(S_{\Delta\cap\mathring{\Lambda}_q}) - W(S_{\Delta\cap\mathring{\Lambda}_q}|S_{\Delta\cap\mathring{\Lambda}_q^c})] \\ \times \exp[-\tfrac{1}{2}aq^d \log q + D_5|\Lambda_q|]\, \rho_\Lambda(S_{\Delta\cap\mathring{\Lambda}_q^c}) \end{aligned} \tag{2.16}$$

By use of Eqs. (2.1) and (2.2), from Eqs. (2.11) and (2.16) it is easy to find the conditions on γ and δ for which the induction hypothesis is fulfilled and then the thesis is proven as soon as conditions (i) and (ii) above are proven.

From Theorem 2.1 one can use, as in Refs. 1 and 3, compactness arguments to prove the existence of DLR measures, thermodynamic limits for the Gibbs measures. The results of Refs. 1 and 3 directly follow, e.g., one can introduce regular DLR measures with support on

$$\{S|E(S) = \sup_{q\geqslant 1}[E(S_{\Lambda_q})/q^d \log q] < +\infty\}$$

and then it can be shown that the regular DLR measures satisfy the superstability estimates (2.4a), (2.4b).

3. COMPLETION OF THE PROOF

In this section we achieve the proof of Theorem 2.1 by proving the following proposition, which for the sake of simplicity will be given for $d = 3$.

Proposition 3.1. For all $S_{\partial\Lambda_q}$ such that $E(S_{\partial\Lambda_q}) < \log q\ |\partial\Lambda q|$, there exist a spin configuration $Z_{\Lambda_q} = \{Z_x\}_{x\in\Lambda_q}$ with the following properties:

(1) $Z_{\partial\Lambda_q} = S_{\partial\Lambda_q}$.

(2) $E(Z_{\Lambda_q}) \leq k_1|\Lambda_q|$ for some constant $k_1 > 0$.

Proof. The idea of the proof is the following. We construct the configuration Z_Λ in such a way that $Z_{\partial\Lambda_r}$, $r < q$, is obtained from $Z_{\partial\Lambda_{r+1}}$ by letting

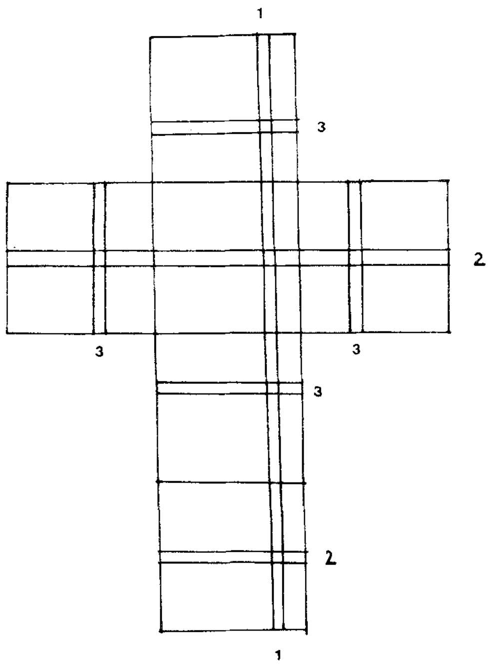

Fig. 1

down the spins of $Z_{\partial\Lambda_{r+1}}$ so that the differences $|S_x - S_y|$, S_x, $S_y \in Z_{\partial\Lambda_r}$, x, y n.n., are about the same as before. This roughly implies that $E(S_{\partial\Lambda_r}) \approx E(S_{\partial\Lambda_{r+1}})$. Moreover, the differences between n.n. spins of $\partial\Lambda_{r+1}$ and $\partial\Lambda_r$ have to be small in order that the global interaction energy between two surfaces not be made too large; but at the same time they need to be large enough to allow the spins to be small after a number of steps of the order of q^δ, $\delta < 1$, in order to satisfy the inequality in part 2.

A counterexample, showing that condition 2 fails if all the spins inside Λ_q are put equal to zero, will be given at the end of this section.

To make rigorous the above idea, we mimic the configuration $S_{\partial\Lambda_q}$ on the boundaries, $\partial\Lambda_r$, $r < q$, by eliminating suitable sites. Now, $Z_{\partial\Lambda_{q-1}}$ may be built starting from $S_{\partial\Lambda_q}$ by eliminating three pairs of internal connected lines of spins on the surface $\partial\Lambda_q$ as in Fig. 1. Such lines, denoted by l_q, are chosen in such a way that their global energy

$$\tilde{E}_q = \sum_{x\in l_q} |S_x|^\alpha + \sum_{\substack{x\in l_q\\ y\in\partial\Lambda_q}}' |S_x - S_y|^\beta \tag{3.1}$$

satisfies

$$\tilde{E}_q \leqslant (k_2/q)E(S_{\partial\Lambda_q}) \tag{3.2}$$

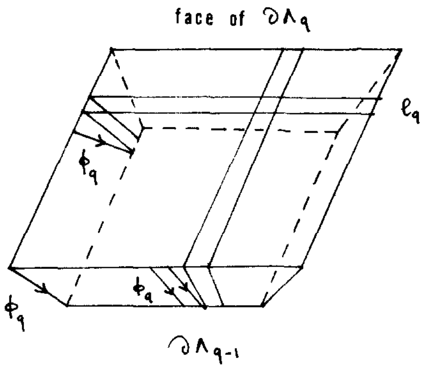

Fig. 2

Since $|\partial\Lambda_q\backslash l_q| = |\partial\Lambda_{q-1}|$, we define the one-to-one map ϕ_q: $\partial\Lambda_q\backslash l_q \to \partial\Lambda_{q-1}$ that associates the boundary of any face of $\partial\Lambda_q$ to the boundary of the corresponding face of $\partial\Lambda_{q-1}$, the corners to the corresponding corners, and such that $\phi_q(x)$, $\phi_q(y)$ are n.n. if x, y are n.n. (See Fig. 2.)

Furthermore, we put

$$Z_x = S_{\phi_q^{-1}(x)}(1 - 1/q^\delta) \tag{3.3}$$

where $\delta < 1$ and will be determined later.

All the above procedure may be iterated $[q^\delta]$ times (here $[k]$ denotes the integer part of k) by putting

$$Z_x = S_y[1 - (q - r)/q^\delta] \tag{3.4}$$

and $y = \phi_q^{-1}[\phi_{q-1}^{-1} \cdots \phi_r^{-1}(x)]$ with an obvious meaning of the symbols. The configuration Z_Λ is completed by putting $Z_x = 0$ for $x \in \Lambda_{q-[q^\delta]-1}$.

With the configuration Z_Λ fixed, we now compute its energy:

$$E(Z_\Lambda) = \sum_{i=0}^{q} E_i + \sum_{i=0}^{q-1} W_i \tag{3.5}$$

where

$$E_i \equiv E(Z_{\partial\Lambda_i}) = \sum_{x\in\partial\Lambda_i} |Z_x|^\alpha + \sum_{x,y\in\partial\Lambda_i}{}' |Z_x - Z_y|^\beta$$

$$W_i = \sum_{\substack{x\in\partial\Lambda_i \\ y\in\partial\Lambda_{i+1}}}{}' |Z_x - Z_y|^\beta \tag{3.6}$$

We have

$$E_r \leqslant E_{r+1} + \sum_{x,y\in\partial\Lambda_r}{}^{*} |Z_x - Z_y|^\beta \tag{3.7}$$

where $\sum^*$ means that we sum over all the n.n. $x, y \in \partial\Lambda_r$ such that $\phi_{r+1}^{-1}(x)$ and $\phi_{r+1}^{-1}(y)$ are not n.n. in $\partial\Lambda_{r+1}$. Furthermore, inequality (3.7) holds since

$$|Z_x - Z_y| \leqslant |Z_{\phi_{r+1}^{-1}(x)} - Z_{\phi_{r+1}^{-1}(y)}|, \qquad \forall x, y \in \partial\Lambda_r$$

The $\sum^*$ may easily be bounded. Denoting by u and v the spins on the lines l_{r+1} that are n.n. of $\phi_{r+1}^{-1}(x)$ and $\phi_{r+1}^{-1}(y)$, we have

$$|Z_x - Z_y|^\beta \leqslant |Z_{\phi_{r+1}^{-1}}(x) - Z_{\phi_{r+1}^{-1}(y)}|^\beta \leqslant c_\beta\{|Z_{\phi_{r+1}^{-1}(x)} - u|^\beta + |u - v|^\beta + |v - Z_{\phi_{r+1}^{-1}(y)}|^\beta\} \tag{3.8}$$

where $c_\beta = \max\{1, 3^{\beta-1}\}$. Then, performing $\sum^*$, we get

$$E_r \leq E_{r+1} + c_\beta \tilde{E}_{r+1} \leqslant E_{r+1}\left(1 + \frac{k_3}{r+1}\right) \tag{3.9}$$

for some constant $k_3 > 0$ [see (3.2)]. Then

$$\sum_{r=q-[q^\delta]}^{q} E_r \leqslant E_q \sum_{r=q-[q^\delta]}^{q} \left(1 + \frac{k_3}{r+1}\right)^{q-r} \leqslant k_4 q^\delta \log q\, |\partial\Lambda_q| \tag{3.10}$$

for some constant $k_4 > 0$.

Let $q - [q^\delta] \leqslant r < q$. Then

$$\begin{aligned} W_r &= \sum_{\substack{x\in\partial\Lambda_r \\ y\in\partial\Lambda_{r+1}}}' |Z_x - Z_y|^\beta \\ &\leqslant c_\beta\left\{\sum (|Z_x - Z_{\phi_{r+1}^{-1}(x)}|^\beta + |Z_{\phi_{r+1}^{-1}(x)} - Z_y|^\beta)\right\} \\ &\leqslant k_5\left\{\sum_{y\in\partial\Lambda_q} |S_y/q^\delta|^\beta + E_{r+1}\right\} \end{aligned} \tag{3.11}$$

for some constant $k_5 > 0$. Hence

$$\sum_{r=0}^{q-1} W_r \leqslant k_5 \sum_{r=0}^{q} E_r + (1 + k_5 q^\delta) \sum_{y\in\partial\Lambda_q} |S_y/q^\delta|^\beta \tag{3.12}$$

because of the following inequality:

$$W_{q-[q^\delta]-1} \leqslant \sum_{y\in\partial\Lambda_q} \left|1 - \frac{[q^\delta]}{q^\delta} S_y\right|^\beta \leqslant \sum_y \left|\frac{S_y}{q^\delta}\right|^\beta \tag{3.13}$$

Combining the estimates (3.12) and (3.10) with the following lemma, we obtain the thesis:

Lemma 3.2. In the hypotheses of Proposition 3.1, the following estimate holds:

$$q^\delta \sum_{y\in\partial\Lambda_q} |S_y/q^\delta|^\beta \leqslant k_6|\Lambda_q| \tag{3.14}$$

for some constant $k_6 > 0$ and $0 \leqslant \delta < 1$.

Proof. Let Γ be a face of $\partial\Lambda_q$ and $g \subset \Gamma$ be a family of parallel lines $\{\Gamma_i\}_{i=1}^n$ with constant distance $[q^{\bar{\delta}}]$, $0 \leqslant \bar{\delta} < 1$; moreover, Γ_1 and Γ_n have a distance from the two parallel corner edges of less than $[q^{\bar{\delta}}]$ (see Fig. 3). Since we have $[q^{\bar{\delta}}]$ ways of choosing such a g, we can find it so that

$$E(g) \leqslant 4E(\Gamma)/q^{\bar{\delta}} \tag{3.15}$$

Here $E(g)$ and $E(\Gamma)$ denote, respectively, the energy of $S_{\partial\Lambda}|_g$ and $S_{\partial\Lambda}|_\Gamma$. Let us denote by Q_i, $i = 1, \ldots, n-1$, the subset of Γ between Γ_i and Γ_{i+1}, including Γ_i, and by Q_0 and Q_n, respectively, the sets between the boundary part parallel to the Γ_i and Γ_1 and Γ_n, including Γ_1 and Γ_n. Then we have for all $x \in Q_i$, $i = 1, \ldots, n-1$,

$$|S_x| \leqslant |S_{x_i}| + \sum_{z,y\in\gamma(x)}' |S_z - S_y| \tag{3.16}$$

and hence

$$|S_x|^\beta \leqslant c_\beta\left[|S_{x_i}|^\beta + |\gamma(x)|^{\rho(\beta)-1} \sum_{z,y\in\gamma(x)}' |S_z - S_y|^\beta\right] \tag{3.17}$$

where $\rho(\beta) = \max\{1, \beta\}$. Here x_i denotes the projection of $x \in Q_i$ on Γ_i, and $\gamma(x)$ and $|\gamma(x)|$ denote, respectively, the straight path from x to x_i and its length (if $x = x_i$, $|\gamma(x)| = 0$). Then

$$\sum_{x\in Q_i} |S_x|^\beta \leqslant c_\beta \sum_{x_i\in\Gamma_i} \sum_x{}^* \left\{|S_{x_i}|^\beta + |\gamma(x)|^{\rho(\beta)-1} \sum_{z,y\in\gamma(x)}' |S_z - S_y|^\beta\right\} \tag{3.18}$$

Here $\sum^*$ denotes the sum over all $x \in Q_i$ with the same projection x_i. Finally,

$$\sum_{x\in Q_i} |S_x|^\beta \leqslant c_\beta\left\{q^{\bar{\delta}} \sum_{x_i\in\Gamma_i} |S_{x_i}|^\beta + q^{\bar{\delta}\rho(\beta)}E(Q_i)\right\} \tag{3.19}$$

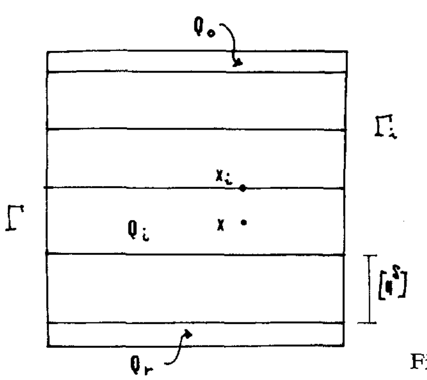

Fig. 3

Analogous estimates may be done for Q_0 and Q_n in term of all $x_i \in \Gamma_1, \Gamma_n$:

$$\sum_{x\in\Gamma} |S_x|^\beta \leqslant k_7\left\{q^{\bar\delta\rho(\beta)}E(\Gamma) + q^{\bar\delta}\sum_{x\in g} |S_x|^\beta\right\} \tag{3.20}$$

with k_7 a positive constant.

The same argument as before may be used to estimate $\sum_{x\in g}|S_x|^\beta$. In fact in any Γ_i we choose a family of points denoted $g_i \equiv \{\Gamma_{ij}\}_{j=1}^{n}$ and such that $E(g_i) \leqslant 4E(\Gamma_i)/q^{\bar\delta}$. Then, following the above argument step by step, we get

$$\sum_{x\in\Gamma_i} |S_x|^\beta \leqslant k_7\left\{q^{\bar\delta\rho(\beta)}E(\Gamma_i) + q^{\bar\delta}\sum_{x\in g_i} |S_x|^\beta\right\} \tag{3.21}$$

and so

$$\sum_{x\in g} |S_x|^\beta \leqslant k_7\left\{q^{\bar\delta\rho(\beta)}E(g) + q^{\bar\delta}\sum_i\sum_{x\in g_i} |S_x|^\beta\right\} \tag{3.22}$$

Let $\bar g = \cup_i g_i$; we have in virtue of (3.15)

$$E(\bar g) \leqslant 4\sum_i \frac{E(\Gamma_i)}{q^{\bar\delta}} \leqslant 16\,\frac{E(\Gamma)}{q^{2\bar\delta}} \tag{3.23}$$

Then, if $x \in \bar g$,

$$|S_x| \leqslant |16/q^{2\bar\delta}|^{1/\alpha}E(\Gamma)^{1/\alpha}$$

and since they are in number not exceeding $4q^{2(1-\bar\delta)}$, we finally get, for some $k_8 > 0$,

$$\sum_{x\in\bar g} |S_x|^\beta \leqslant 4q^{2(1-\bar\delta)}(16)^{\beta/\alpha}E(\Gamma)^{\beta/\alpha}q^{-2\bar\delta(\beta/\alpha)}$$

$$\leqslant k_8 q^{2(\beta/\alpha)(1-\bar\delta)+2(1-\bar\delta)}(\log q)^{\beta/\alpha} \tag{3.24}$$

Then we can fix $\bar\delta < 1$ such that

$$\sum_{x\in\bar g} |S_x|^\beta \leqslant k_8 q^{\eta+2(1-\bar\delta)} \tag{3.25}$$

with $0 < \eta < 1$.

In virtue of the estimates (3.20), (3.22), (3.15), (3.24), and (3.25), we get

$$\sum_{x\in\Gamma} |S_x|^\beta \leqslant k_9\{q^{\bar\delta\rho(\beta)}E(\Gamma) + q^{2+\eta}\}$$

and summing over all the faces, we obtain

$$q^{\delta(1-\beta)} \sum_{x\in\partial\Lambda_q} |S_x|^\beta \leqslant k_{10}\{q^{\delta(1-\beta)+\bar\delta\rho(\beta)+2}\log q + q^{2+\eta+\delta(1-\beta)}\}$$

$$\leqslant k_{11}q^3$$

for a suitable choice of $\delta < 1$, so that the lemma is proven.

Remark 1. If $\beta \leqslant 1$, we can choose $\delta = 0$, that is, it is possible to obtain a "good" configuration Z_{Λ_q} such that $Z_x = 0$ if $x \notin \partial\Lambda_q$.

Remark 2. If $d \neq 3$ the above argument should be repeated $d - 1$ times instead of twice. The proof is then completely analogous.

Minor modifications allow us to prove statements (i) and (ii) of Section 2 from Proposition 3.1. By changing constants we can write q^d, q^{d-1} instead of $|\Lambda_q|$, $|\partial\Lambda_q|$. Furthermore, if we consider a configuration S'_{Λ_q} which differs as in (ii) from Z_{Λ_q}, i.e.,

$$|S_x' - Z_x| \leqslant 1, \qquad x \in \Lambda_q$$

then, by the inequality

$$E(S'_{\Lambda_q}) \leqslant c_\beta E(Z_{\Lambda_q}) + k_{12} q^d$$

valid for suitably large k_{12}, we obtain the result.

We complete this section by giving an example of a configuration $S_{\partial\Lambda_q}$ with $E(S_{\partial\Lambda_q}) \leqslant q^{d-1}\log q$ such that the configuration S_{Λ_q} obtained by putting $S_x = 0$ for $x \in \mathring{\Lambda}_q$ has energy $E(S_{\Lambda_q}) > cq^{d+\epsilon}$ for ϵ sufficiently small and $c > 0$.

Let $\Gamma = \{x \in \Lambda q | x^1 = q\}$ be a face of Λ_q and $x_0 = (q, 0, \ldots, 0)$ its center. We consider a configuration S_{Λ_q} such that

(1) $S_x = \bar{S}(1 - k/R)$ if $x \in \Gamma$ and $|x - x_0| = k \leqslant R \leqslant q$.

(2) $S_x = 0$ otherwise.

Here $\bar{S}$ and R, R integer, are positive constants to be fixed later. Observe now that there exist c_1, c_2, c_3, c_4 so that

$$E(S_{\partial\Lambda_q}) \leqslant c_1 \sum_{0}^{R}{}_k\, k^{d-2}\left[\bar{S}^\alpha\left(1 - \frac{k}{R}\right)^\alpha + \left(\frac{\bar{S}}{R}\right)^\beta\right]$$
$$\leqslant c_3(R^{d-1}\bar{S}^\alpha + R^{d-1-\beta}\bar{S}^\beta)$$

$$E(S_{\Lambda_q}) \geqslant c_2 \sum_{0}^{R}{}_k\, k^{d-2}\bar{S}^\beta\left(1 - \frac{k}{R}\right)^\beta \geqslant c_4 R^{d-1}\bar{S}^\beta$$

For the example in which we are interested, it is sufficient to be able to choose $\bar{S}$ and R so that

$$\begin{aligned} R^{d-1}\bar{S}^\alpha + R^{d-1-\beta}\bar{S}^\beta &\leqslant c_5 q^{d-1} \\ R^{d-1}\bar{S}^\beta &\geqslant c_6 q^{d+\eta} \end{aligned} \tag{3.26}$$

for suitable c_5, c_6, and $\eta > 0$. If we put

$$R = q^\epsilon, \qquad 0 \leqslant \epsilon \leqslant 1 \tag{3.27}$$

it is easy to see that (3.26) can be satisfied if

$$\frac{1 + \eta}{\beta} \leqslant \epsilon \leqslant \frac{1 - [(d + \eta)/(d - 1)]\alpha/\beta}{1 - \alpha/\beta} \tag{3.28}$$

It is now clear that conditions (3.27) and (3.28) can both be satisfied for a suitable $\eta > 0$ only if $\beta > 1$ (in agreement with remark 1) and d is large enough.

4. CONCLUDING REMARKS

In proving Proposition 3.1 a central role is played by Lemma 3.2, which gives a way of bounding $\sum S_x{}^\beta$ in terms of $\sum' |S_x - S_y|^\beta$. This recalls the classical Sobolev inequalities, which in our context read: let $\{S_x | x \in \mathbb{Z}^\nu, S_x = 0 \text{ for } x \in \Gamma^c, \Gamma \text{ bounded set}\}$ be a spin configuration. Then

$$\left(\sum_x |S_x|^\beta\right)^{1/\beta} \leqslant \left(\sum_{x,y} |S_x - S_y|^\gamma\right)^{1/\gamma}$$

where $1/\gamma = 1/\beta + 1/\nu$.

Unfortunately, inequalities such as the above do not seem to work directly in our case because we need to bound the l_β norm of S in terms of the same l_β norm of the gradient, and to get such bound the convexity inequality gives bad estimates for our purposes. The idea of Lemma 3.2 is to use the convexity inequality as little as possible and along suitable low-energy paths.

Theorem 2.1 has been proven under the assumption that $\alpha > 0$. This plays an essential role in our method (see Section 3). If one assumes $\alpha = 0$, $\beta = 2$, then it is well known[4] that for $d = 1, 2$ no limiting state exists, while for $d = 3$ one does. In the case $\alpha = 0$ one could look only at the random variables which are the differences between n.n. spins. This leads to a new infinite-spin system, where now $\alpha \neq 0$ but in each plaquette the sum of the new spins should be zero. This also cannot be treated straightforwardly with our method, because it corresponds somehow to $\beta = \infty$.

The extension of our results to infinite range would also be of interest and does not follow straightforwardly from our considerations.

Another interesting extension is connected with the lattice spacing going to zero. One should get limiting states with support on trajectories, whose continuity properties depend on the way we choose the interaction dependence on the lattice spacing.

We finally mention that Theorem 2.1 allows us to extend the proof in Ref. 5 of the existence of the dynamics for anharmonic crystals. Namely, in that paper it was assumed that $\alpha \geqslant \beta$ (in our notation) only to ensure that the equilibrium state exists and satisfies the superstability estimates.

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